# Divisibility and a weak ascending chain condition on principal ideals

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# Notations and Assumptions

- $\mathbb{N}: \{1, 2, \ldots\}$.
- $\mathbb{N}_0: \{0, 1, 2, \ldots\}$.
- We let $v_p$ to denote the $p$-adic valuation on $\mathbb{Z}^*$, namely, $v_p(a)$ is the largest positive integer $n$ such that $p^n \mid a$.

# Definition: Monoid

**$(M, *)$ is a monoid if ...**

For any $a, b \in M$, we can define $a * b \in M$, which satisfies $(a * b) * c = a * (b * c)$ (assoicativity).
Also, there exists $e \in M$ such that $e * a = a * e = a$ for any $a \in M$.

We will further assume that all monoids are commutative ($a * b = b * a$) and cancellative ($a * c = b * c$ implies $a = b$).

**Definition: submonoid**

For a monoid $M$, a submonoid of $M$ is a subset that form a monoid itself under the operation of $M$.

# Definitions: Atomicity and ACCP

## Definition: atomicity

A monoid is atomic if every (nonunit) element can be factored into irreducibles.

## Definition: principal ideal

A principal ideal of $(M, *)$ is a subset of $M$ with the form $b * M = \{b * m : m \in M\}$ for some $b \in M$.

## Definition: ACCP

A monoid satisfies the ACCP if every ascending chain of principal ideals eventually stabilizes.

## For integral domains . . .

We say an integral domain $R$ is atomic (resp., satisfies the ACCP) if the multiplicative monoid of $R$ is atomic (resp., satisfies the ACCP).

# Examples

## Example: nonnegative integers

$(\mathbb{N}_0, +)$ is a monoid that is both atomic and satisfies the ACCP.

## Example: multiplicative monoid

$(\mathbb{Z} \setminus \{0\}, \cdot)$ is both atomic and satisfies the ACCP. Thus, the integral domain $\mathbb{Z}$ is both atomic and satisfies the ACCP.

## Example: a submonoid of $(\mathbb{Q}_{\geq 0}, +)$

$\left\langle \frac{1}{p} : p \text{ is prime} \right\rangle$ is both atomic and satisfies the ACCP.

# More Examples

**Example: an atomic monoid without the ACCP**

- $p_1 = 3, p_2 = 5, \ldots$ be the sequence of odd primes.
- $$M = \left\langle \frac{1}{2^k p_k} : k \in \mathbb{N} \right\rangle \subseteq \mathbb{Q}_{\geq 0}$$
- $\left(\frac{1}{2^k} + M\right)_{k=1}^{\infty}$ is an ascending chain of principal ideals that does not stabilize (note that $\frac{1}{2^k} = p_k \cdot \frac{1}{2^k p_k} \in M$).

# Monoid Rings

**Definition: monoid ring**

We use $R[M]$ to denote the monoid ring with coefficients in $R$ and exponents in $M$, that is, all finite linear combinations of $x^m$ with $m \in M$, equipped with the multiplication $x^{m_1} \cdot x^{m_2} = x^{m_1+m_2}$.

**Proposition**

$R[M]$ is an integral domain iff $R$ is an integral domain and $M$ is torsion-free ($na = nb$ implies $a = b$).

**Proposition**

For a field $F$, we have $F[M]$ satisfies the ACCP provided that $M$ is torsion-free and satisfies the ACCP.

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# History

- Every monoid satisfying ACCP is atomic.
- $$M = \left\langle \frac{1}{2^k p_k} : k \in \mathbb{N} \right\rangle.$$
- Cohn's assertion: every atomic domain satisfies the ACCP.
- Grams: first counterexample.
- Producing such examples is challenging.

# First Example

## Grams' example

- Fix a field $F$.
- $$M = \left\langle \frac{1}{2^k p_k} : k \in \mathbb{N} \right\rangle.$$
- $S := \{f \in F[M] : f(0) \neq 0\}$.
- Grams proved that $F[M]_S$ is atomic but does not satisfy the ACCP.

# Other Examples Revisit

## Zaks' example

Back in the eighties, Zaks proved that two atomic domains do not satisfy the ACCP.

- One is of the form $F[M]_S$.
- The other is a monoid ring.

## Roitman's example

Roitman constructed the first atomic domain $R$ such that $R[x]$ is not atomic; hence $R$ does not satisfy the ACCP.

## Boynton and Coykendall's example

In 2019, Boynton and Coykendall constructed an atomic domain without ACCP based on integer-valued polynomials.

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# Motivation

- Grams' example looks like $F[M]_S$ for a certain monoid $M$ and $S = \{f \in F[M] : f(0) \neq 0\}$.
- Question: can we find more such $M$'s?
- Partial answer: yes, using greatest divisor submonoid.

# Definition: Greatest Divisor Submonoid

Let $N$ be a submonoid of $M$.

## Definition: greatest divisor

For each $m \in M$, a greatest divisor of $m$ in $N$ is an element $d \in N$ satisfying the following two conditions:

- $d \mid_M m$.
- If $d' \mid_M m$ for some $d' \in N$, then $d' \mid_M d$.

## Definition: greatest divisor submonoid

- We say $N$ is a greatest divisor submonoid of $M$ if any $m \in M$ has a greatest divisor in $N$.
- In case $M$ is reduced (the only unit is $0$), the greatest divisor is unique and is denoted by $\mathrm{gd}_N(m)$.

# Grams-Like Construction

**Definition: positive monoid and Puiseux monoid**

- A positive monoid is a submonoid of $\mathbb{R}_{\geq 0}$.
- A Puiseux monoid is a submonoid of $\mathbb{Q}_{\geq 0}$.

**Grams-Like Construction Theorem (Gotti-L., 2021)**

Let $F$ be a field, and let $M$ be a positive monoid. Also, let $N$ be a submonoid of $M$ satisfying the following conditions:

- $N$ is a valuation greatest-divisor submonoid of $M$, and
- For each $m \in M$, there exists $L \in \mathbb{N}$ such that $m - \mathrm{gd}_N(m)$ cannot be factored into more than $L$ nonzero elements.

Then $F[M]_S$ is atomic, where $S = \{f \in F[M] : f(0) \neq 0\}$.

# Atomization: Definition

## Definition: atomization

- Fix a Puiseux monoid $N = \langle a_i/b_i : i \in \mathbb{N} \rangle$, where $a_i, b_i \in \mathbb{N}$ with $\gcd(a_i, b_i) = 1$.
- Pick a sequence of primes $(p_k)_{k=1}^{\infty}$ such that $\gcd(p_i, a_i) = 1$ and $\gcd(p_i, b_j) = 1$ for any $i, j \in \mathbb{N}$.
- Then we say

$$M := \left\langle \frac{a_k}{b_k p_k} : k \in \mathbb{N} \right\rangle$$

is an atomization of $N$.

# Atomization: Application

**Proposition (Gotti-L., 2021)**

If $M$ is an atomization of a valuation monoid $N$, then $N$ is a greatest divisor submonoid of $M$ and $m - \mathrm{gd}_N(m)$ has only 1 factorization (in $M$) for any $m \in M$.

*Condition 2: For each $m \in M$, there exists $L \in \mathbb{N}$ such that $m - \mathrm{gd}_N(m)$ cannot be factored into more than $L$ nonzero elements.*

**Example: Grams' example**

- Pick $N = \langle 1/2^k : k \in \mathbb{N} \rangle$ and $(p_k)_{k=1}^{\infty}$ to be the sequence of odd primes. Note that $N$ is a valuation monoid.
- $$M = \left\langle \frac{1}{2^k p_k} : k \in \mathbb{N} \right\rangle$$
  is an atomization of $N$. Thus, $F[M]_S$ is atomic.

# Imperfections

- Grams' construction $F[M]_S$ is not the whole story.
- $\mathbb{N}_0$ is a greatest divisor submonoid of $M = \langle 2^k/3^k : k \in \mathbb{N} \rangle$ and satisfies the condition in Grams-Like Construction Theorem. However, it cannot be built from atomization.
- Greatest divisor submonoid itself has no relation with either atomicity or ACCP.
- Is there some more essential things related to atomicity that we can extract from this structure?

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# Motivation

Very rough sketch of Grams' proof that $F[M]_S$ is atomic:

- Every element $f$ can be written in the form

$$f = x^{\frac{k}{2^n}} \cdot f_0.$$

- There exists $N \in \mathbb{N}$ such that $f_0$ cannot be written as the product of $N$ nonunits.
-
$$x^{\frac{k}{2^n}} = \left(x^{\frac{1}{2^n p_n}}\right)^{kp_n}.$$

Also, in some other atomic monoids that do not satisfy the ACCP, similar pattern holds. Inspired by these, we extract their common properties out and form the definition of weak-ACCP.

# Definition

## Definition: weak-ACCP

A monoid $M$ is weak-ACCP if every nonempty finite subset $S$ of $M$ has a common divisor $d \in M$ such that

- $d$ can be factored into the product of atoms, and
- there exists $s \in S$ such that any ascending chain of principal ideals starting at $s/d$ eventually stabilizes.

## For integral domains ...

An integral domain is weak-ACCP if its multiplicative monoid is weak-ACCP (as in the case of atomic/ACCP).

# Weak-ACCP is Weak ACCP

- ACCP $\Rightarrow$ weak-ACCP $\Rightarrow$ atomic.
- None of the converse implication holds.
- Weak-ACCP $\not\Rightarrow$ ACCP:

$$M = \left\langle \frac{1}{2^k p_k} : k \in \mathbb{N} \right\rangle.$$

- Atomic $\not\Rightarrow$ weak-ACCP:
- $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$
- $$M = \left\langle \frac{1}{p_k p_{k+2}} : k \in \mathbb{N} \right\rangle.$$

# Why Do We Care?

A natural question is why should we care about the property of being weak-ACCP. Here are some reasons:

1. It helps us to find new examples of atomic domains without ACCP.
2. It has some nice properties that atomicity does not guarantee.

**Proposition (Gotti-L., 2022)**

If an integral domain $R$ is weak-ACCP, then $R[x]$ is also weak-ACCP.

3. Most known atomic domains without ACCP are actually weak-ACCP.
4. In many cases the difficulty of proving that certain domain is atomic is about the same as the difficulty of proving it is weak-ACCP.
5. Weak-ACCP allows us to get a better understanding of the structure of many integral domains.

# Counter-Examples Re-Revisit

## Zaks' example

*Back in the eighties, Zaks proved that two atomic domains do not satisfy the ACCP.*

Both of them are weak-ACCP.

## Roitman's example

*Roitman constructed the first atomic domain $R$ such that $R[x]$ is not atomic; hence $R$ does not satisfy the ACCP.*

It is NOT weak-ACCP.

## Boynton and Coykendall's example

*In 2019, Boynton and Coykendall constructed an atomic domain without ACCP using pullback constructions.*

It is weak-ACCP.

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# Propositions

## Proposition (Gotti-L., 2022)

If an integral domain $R$ is weak-ACCP, then $R[x]$ is also weak-ACCP.

## Proposition (Gotti-L., 2022)

Let $M$ be a positive monoid. Also, let $N$ be a submonoid of $M$ satisfying the following conditions:

- $N$ is a valuation greatest-divisor submonoid of $M$, and
- For each $m \in M$, there exists $L \in \mathbb{N}$ such that $m - \text{gd}_N(m)$ cannot be factored into more than $L$ nonzero elements.

Then $M$ is weak-ACCP.

## Proposition (Gotti-L., 2022)

Fix a field $F$. For a positive monoid $M$, if $M$ is weak-ACCP, then $F[M]_S$ is also weak-ACCP, where $S$ is the multiplicative subset $\{f \in F[M] : f(0) \neq 0\}$.

# Finding Weak-ACCP Monoids

## Weak-ACCP Producing Theorem (Gotti-L., 2022)

Let $(a_i)_{i=1}^{\infty}, (b_i)_{i=1}^{\infty}$ be two sequences of positive integers with $\gcd(a_i, b_i) = 1$ for every $i \in \mathbb{N}$ satisfying the following conditions:

- $a_n \mid a_{n+1}$ for every $n \in \mathbb{N}$.
- For each prime $p$, there exists $n_p \in \mathbb{N} \cup \{\infty\}$ such that $v_p(b_n) \geq v_p(b_{n-1})$ for every $1 < n \leq n_p$, and $v_p(b_n) \leq v_p(b_{n-1})$ for every $n > n_p$ (i.e. the sequence $(v_p(b_i))_{i=1}^{\infty}$ is unimodal).

Let $M = \langle a_i/b_i : i \in \mathbb{N} \rangle$. Then the following conditions are equivalent:

1. $M$ is atomic;
2. $M$ is weak-ACCP;
3. $M$ is finitely generated or $M$ has infinitely many atoms.

# Examples of Weak-ACCP Monoids/Domains

**Example: Grams' domain**

- Grams' monoid
$$M = \left\langle \frac{1}{2^k p_k} : k \in \mathbb{N} \right\rangle$$
is weak-ACCP.
- Grams' domain $F[M]_S$ is also weak-ACCP, whence atomic.

# More Examples

## Another known example

- Coprime integers $a, b$ such that $2 \leq a < b$.
- $$M = \left\langle \frac{a^k}{b^k} : k \in \mathbb{N} \right\rangle$$
  is weak-ACCP.
- $M$ does not satisfy the ACCP:
  $$b \cdot \frac{a^i}{b^i} = b \cdot \frac{a^{i+1}}{b^{i+1}} + (b-a) \cdot \frac{a^i}{b^i}.$$
- $F[M]_S$ is weak-ACCP but does not satisfy the ACCP.

# More Examples

## Previously unknown example

- Primes $p_1 < p_2 < \cdots$
- $$M = \left\langle \frac{1}{p_k p_{k+1}} : k \in \mathbb{N} \right\rangle$$
  is weak-ACCP.
- $M$ does not satisfy the ACCP:
  $$\frac{1}{p_i} = \frac{1}{p_{i+1}} + \frac{p_{i+1} - p_i}{p_i p_{i+1}}.$$
- $F[M]_S$ is weak-ACCP but does not satisfy the ACCP.

# More Examples

## Even more

- Primes $p_1 < p_2 < \cdots$
- $$M_1 = \left\langle \frac{1}{p_k p_{k+1}^2 p_{k+2}} : k \in \mathbb{N} \right\rangle .$$
$$M_2 = \left\langle \frac{1}{\prod_{i=k}^{2k} p_i} : k \in \mathbb{N} \right\rangle .$$
- $F[M_i]_S$ are weak-ACCP but does not satisfy the ACCP.

# Imperfections Resolved

- *Grams' construction $F[M]_S$ is not the whole story.*
  Various historical examples are weak-ACCP.
- $\mathbb{N}_0$ *is a greatest divisor submonoid of* $M = \langle 2^k/3^k : k \in \mathbb{N} \rangle$ *and satisfies the condition in Grams-Like Construction Theorem. However, it cannot be built from atomization.*
  We can show that $M$ is weak-ACCP using Weak-ACCP Producing Theorem directly.
- *Greatest divisor submonoid itself has no relation with either atomicity or ACCP.*
  Weak-ACCP lies between atomicity and ACCP.

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Thank You!